HEAT TRANSFER IN TURBULENT MOTION

(PERENOS TEPLA V TURBULENTNOM DVIZHENII)

PMM Vol.24, No.4, 1960, pp. 637-646

L.H. LOITSIANTSKY
(Leningrad)

(Received 6 April 1960)

1. A short survey of the development of the theory of heat transfer in turbulent flow. Prandtl [1] in 1910 and later in 1928, and independently Taylor [2] in 1916, proposed a "two-layer" - laminar sub-layer and turbulent core - theory of heat transfer for Prandtl numbers different from unity, when the Reynolds analogy in its classical form becomes inapplicable. The application of this theory proved to be limited to a narrow range of Prandtl numbers in the neighborhood of unity.

To Karman [3,4] belongs the credit of giving in 1934-39 the first theory of turbulent heat exchange which takes into account the interaction of the molecular and molar processes in the fluid. The range of Prandtl numbers for which the theory gave agreement with experimental results was noticeably increased. Rejecting the "two-layer" scheme Karman placed (Fig. 1) between the laminar sub-layer 1 and the turbulent core 2 a new flow, a transition region, or, as it is often called, a buffer region 3, representing in this manner more than half of the old laminar sub-layer and part of the turbulent core. In terms of universal variables

$$\eta = \frac{y v_*}{\nu}, \qquad \varphi = \frac{u}{v_*}, \qquad v_* = \sqrt{\frac{\tau_w}{\rho}} \tag{1.1}$$

where τ_w is the shearing stress on a rigid wall, u is the longitudinal velocity, y is the coordinate normal to the surface, and ν and ρ are respectively the kinematic viscosity and the density of the fluid; this transition layer appears for $5 < \eta \leqslant 30$. The distribution of velocities of the flow is determined by the system of equations

$$\begin{aligned} &\varphi = \eta \qquad \text{for } 0 \leqslant \eta \leqslant 5 \\ &\varphi = 11.5 \log \frac{\eta}{5} + 5 \qquad \text{for } 5 \leqslant \eta \leqslant 30 \\ &\varphi = 5.75 \log \eta + 5.5 \qquad \text{for } 30 \leqslant \eta \end{aligned} \tag{1.2}$$

For the distribution of heat output from the wall, Karman introduced the Stanton number

$$S_m = \frac{N_m}{R_m P} = \frac{q_w}{\rho U_m c_p \theta_m} \qquad \left(N_m = \frac{q_w d}{\lambda \theta_m}, \quad R_m = \frac{U_m d}{\nu}, \quad P = \frac{\mu c_p}{\lambda}\right) \tag{1.3}$$

expressing in a most convenient fashion the simple non-dimensional numbers of Nusselt N_m, Reynolds R_m and Prandtl P.

Here q_w is the rate of heat flux passing through the surface of the body, U_m is the velocity of the unperturbed flow distant from the surface of the body, θ_m is the difference between the temperature of the surface and that of the unperturbed flow, μ and λ are, respectively, dynamic coefficients of viscosity and heat conductivity and c_p is the specific heat. By means of simple computations Karman determined the formulas

$$\frac{1}{S_m} = \frac{2}{c_f} + \left(\frac{2}{c_f}\right)^{1/2} g(P), \qquad c_f = \frac{2\tau_w}{\rho U^2} \tag{1.4}$$

$$g(P) = 5\left\{(P-1) + 2.3 \log\left[1 + \tfrac{5}{6}(P-1)\right]\right\}$$

An analogous computation carried out for the two-layer theory of Prandtl-Taylor would give the following expression for the function $g(P)$:

$$g(P) = 8.7(P-1) \tag{1.5}$$

Comparing the results of the computations (Fig. 2) from Formulas (1.4) and (1.5) with experimental material of Eagle and Ferguson [5], Karman showed the advantage of his theory. Whereas the formula of Prandtl-Taylor (curve numbered 1) gave indication for $P = 2$ of deviating from the experimental results of Eagle and Ferguson for indicated values of the number R, Karman's formula (1.4) (curve 2) showed applicability up to $P = 15$. Karman himself observed the increasing divergence of his theory with the experimental curve of Ditus and Boelter [6], beginning with $P = 15$. The next development in the theory and in the experiments showed that the reason for this divergence appeared to be a deficiency in Karman's theory – not taking into account the influence of turbulent pulsations in the neighborhood of the walls. The presence of such pulsations and their connection with the transfer of heat were confirmed in the experiments considered in the work of Del-Nunzio [7], going back to 1930-31. The experiments of Reichardt, Mutchfelder and Reichardt and Shuch [8] show the influence of the pulsations on the profile of the mean velocity near the surface of the body. This influence leads to a small but calculably noticeable deviation in the velocity curve from the purely laminar case, represented first by (1.2) (we see this again in Fig. 4). Analogous results were obtained in the later experiments of Laufer, Schubauer and Klebanoff [9]. Here in the Soviet Union Kapitsa, Landau and Levich [10]

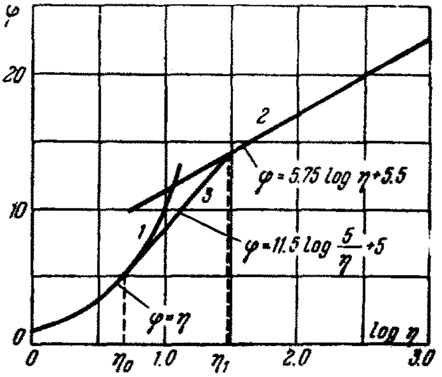

Fig. 1.

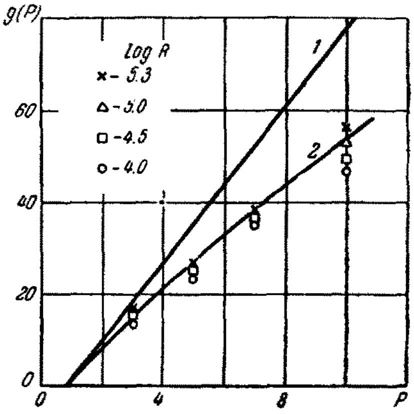

Fig. 2.

turned their attention to this situation. Levich put it into the scheme of his theory of heat-mass exchange.

The law of the decrease of turbulent shearing stress on approaching the wall,

$$\tau_t = -\rho\overline{u'v'} = \rho\varepsilon\frac{du}{dy} \tag{1.6}$$

a question of great importance in the theory of turbulent heat-mass exchange, has been considered by a number of authors. Reichardt [11], starting out from a consideration of the order of decrease of the pulsations ($u' \approx y$, $v' \approx y^2$), but not taking into account the influence of the correlation coefficient between them, arrived at the cubic law $\epsilon \approx y^3$; this law was employed in the investigation of Lin, Moulton and Putnam [12].

In 1943, Levich [10], by considering the rate of decrease of the velocity pulsations, and also by using a supplementary hypothesis on the independence of the period of the pulsations from the distance from the wall, came to the conclusion that the coefficient of turbulent displacement, ϵ, must decrease as one approaches the wall as the fourth power of the distance from the wall. As can be seen from the formula (1.6), one cannot immediately deduce the rate of decrease of ϵ because the velocity pulsations are correlated, and the value of the correlation coefficient is rather difficult to determine. An argument given by Elrod [13], who determined that the power in the law of the decrease of turbulent viscosity cannot be less than four, proved to be erroneous. Deissler discovered the law of the fourth power [14]. On the basis of a collection of wide-ranging experimental material on heat-mass transfer and also some partly empirical considerations he derived the formula

$$\frac{\varepsilon}{\nu} = n^4\eta^4, \qquad n = 0{,}124 \tag{1.7}$$

The question of heat transfer for very small values of the Prandtl number is taken up in the work of Martinelli [15] and Lyon [16]. A systematic presentation of the results in this area can be found in a monograph of Kutateladze, Borizhanskii, Novikov and Fed'inskii [17].

2. A generalization of the hypothesis of localness for the case of interaction between molar and molecular exchange.

The hypothesis of localness of turbulent mixing, based on the semi-empirical theory of Prandtl-Karman, which disregards the influence of the molecular viscosity on the molar transfer of momentum, can be successfully applied also in the case when molecular and molar processes quantitatively compare with one another and when it is no longer allowable to disregard their interaction.

In the work of the author [18] this interaction was taken into account. With the idea in mind of producing a semi-empirical theory of turbulence the concept of a local Reynolds number was introduced. That number was at that time defined as the ratio of the kinematic coefficient of turbulent mixing to the kinematic viscosity, that is the ratio ϵ/ν where, as one could deduce from the content, ϵ is understood to be the turbulent mixing coefficient which does not depend explicitly on the magnitude of the molecular viscosity coefficient; it is characterized by the usual formulas of the semi-empirical theory of Prandtl and Karman. The role of the characteristic length in the local Reynolds number is the "mixing length" l, and the characteristic velocity U is the magnitude of the relative velocity between adjacent layers of the mean turbulent motion $U = l(du/dy)$. This leads to the following quantitative definition of the local Reynolds number:

$$R = \frac{Ul}{\nu} = \frac{l^2}{\nu}\frac{du}{dy} \tag{2.1}$$

In the same paper, it was proved that the local analog of the Euler number is

$$E = \frac{\tau}{\rho U^2} = \frac{\tau}{\rho l^2 (du/dy)^2} \tag{2.2}$$

where τ, the actual turbulent shear stress, containing in itself both the molecular and the molar exchange of momentum, must be a function of the local Reynolds number

$$\frac{\tau}{\rho l^2 (du/dy)^2} = F\left(\frac{l^2}{\nu}\frac{du}{dy}\right) \tag{2.3}$$

Attempts to define the form of the function F were at the time not successful. In [19] the relationship in Equation (2.3) was written in the form

$$\tau = \mu \frac{du}{dy} f(R), \qquad f(R) = RF(R) \tag{2.4}$$

In this case R was understood as the expression (2.1) and, more important from the point of view of the hypothesis of localness, the expression

$$R = \frac{\varkappa^2}{\nu} \frac{(du / dy)^3}{(d^2u / dy^2)^2} \tag{2.5}$$

The form of dependence of $f(R)$ in that paper came from purely intuitional considerations, taking the form of a segment of an equilateral hyperbola outside the laminar sub-layer and a segment of a straight line to the axis of R in the sub-layer itself.

Considering the case of turbulent motion which is established parallel to an infinite, smooth, plane wall, we obtain

$$l = \varkappa y, \qquad \tau = \mu \frac{du}{dy}\left(1 + \frac{\varepsilon}{\nu}\right) = \mu \frac{du}{dy} f(R) = \text{const} = \tau_w \tag{2.6}$$

Thus we shall have

$$1 = \frac{d\varphi}{d\eta} f(R), \qquad R = \varkappa^2 \eta^2 \frac{d\varphi}{d\eta} \tag{2.7}$$

Near to the smooth wall where $\psi = \eta$, according to (2.7) and the formula of Deissler (1.7), we get

$$f(R) \doteq 1 + n^4\eta^4 \doteq 1 + \alpha R^2 \qquad \left(\alpha = \left[\frac{n}{\varkappa}\right]^4\right) \tag{2.8}$$

For $\kappa = 0.4$ and $n = 0.124$ we have $a = 0.0092$.

Far from the wall it is possible to assume that the usual molecular viscosity does not influence the molar turbulent processes, so that the correct asymptotic (for large R) relation

$$f(R) \sim R + 1 \tag{2.9}$$

expresses a simple superposition of the molecular and molar friction.

The establishment in this fashion of the asymptotic expressions (2.8) and (2.9), which correspond to small and large R, can be taken as the basis of the approximate semi-empirical definition of the characteristic function of the interaction of the molecular and molar mixing $f(R)$ for the whole range of R from 0 to ∞. Accordingly we observe that the function $F(R)$ introduced above must take the following form for small and large values of the local Reynolds number by its definition (2.3) and because of the established equalities (2.8) and (2.9):

$$F(R) \doteq \frac{1}{R} + \alpha R, \qquad F(R) \sim 1 + \frac{1}{R} \tag{2.10}$$

We start from the quantitative analogy between the local Reynolds number dependence of the quantity F, which represents the local Euler number, according to (2.3) and the dependence of the coefficient of resistance of a smooth tube on the Reynolds number of the main flow. Thus it follows from (2.10) that the function

$$F_1(R) = F(R) - \frac{1}{R} \tag{2.11}$$

will be together with all of its derivatives a bounded, continuous, smooth and monotonic function of R in the interval $0 < R < \infty$. We note that since as $R \to \infty$ the function $F_1(R)$ tends to a limiting value equal to unity, it is possible to assume for large R

$$\frac{dF_1}{dR} = k(1 - F_1)^r \tag{2.12}$$

where k and r are as yet undetermined positive numbers.

Collecting the successive derivatives with respect to R from both sides of Equation (2.12) and using the boundary conditions imposed on the derivatives we realize that the relation $r \geqslant 1$ must hold.

Integration of Equation (2.12) gives

$$F_1 = 1 - [C + k(r-1)\,R]^{-\frac{1}{r-1}} \tag{2.13}$$

If it is required that the function $F_1(R)$ satisfy both of the limiting conditions (2.10), then the arbitrary constants C and k must be given by

$$C = 1, \qquad k = \alpha \tag{2.14}$$

and so

$$F_1(R) = 1 - [1 + \alpha(r-1)R]^{-\frac{1}{r-1}} \tag{2.15}$$

and also

$$F(R) = 1 + \frac{1}{R} - [1 + \alpha(r-1)R]^{-\frac{1}{r-1}} \tag{2.16}$$

And so in agreement with Formula (2.4) we obtain the following expression for the desired characteristic function of the interaction:

$$f(R) = 1 + R\{1 - [1 + \alpha(r-1)R]^{-\frac{1}{r-1}}\} \tag{2.17}$$

The constant R in this case remains undetermined.

Using (2.17), we construct an asymptotic expansion for $f(R)$ (for large R)

$$f(R) \sim R + 1 - [\alpha(r-1)]^{-\frac{1}{r-1}} R^{-\frac{2-r}{r-1}} \tag{2.18}$$

We recall that the previously assumed condition (2.9) of the independence of the molecular and molar processes of mixing on distance from the solid boundary gave as the leading members of the expansion the first two terms. We conclude from this that the third member of the expansion must contain R to a negative power, so that if we also take into consideration the previously mentioned inequality $r \geqslant 1$, we find

$$1 \leqslant r < 2 \tag{2.19}$$

Returning now to relation (2.17) and inserting into it $r - 1 = 1/s$, requiring $1 < s < \infty$, we obtain finally the required function of the interaction

$$f(R) = 1 + R\left[1 - \left(1 + \frac{\alpha}{s} R\right)^{-s}\right] \tag{2.20}$$

According to contemporary theories of turbulent mixing there is no basis for the rational determination of the constant s. However, for the determination of the quantity a it was necessary to employ experimental materials for heat-mass transfer for very large values of the Prandtl number or the Schmidt number. Analogously it is correct also to determine s in this manner. It is necessary to compare the solution with experimental material for very small values of the Prandtl or the Schmidt number.

We notice that the right-hand side of (2.20) depends weakly on s. For the smallest integer $s = 2$, we have

$$f(R) = 1 + R - \frac{R}{(1 + {}^1/_2 \alpha R)^2} \tag{2.21}$$

and with $s = \infty$ we obtain the exponential law

$$f(R) = 1 + R\,(1 - e^{-\alpha R}) \tag{2.22}$$

The full curves of Fig. 3 represent two forms of the function $f(R)$, Curve 1 corresponding to Formula (2.21) and Curve 2 to (2.22). As can be seen, they differ slightly from each other. On the same figure the dotted curves 3 and 4 correspond to Formulas (2.8) and (2.9), expressing the behavior of the characteristic function for very small and very large values of R.

3. Extension of the Karman theory to other values of the Prandtl number. We will retain the general scheme of Karman in partitioning the flow into three layers, with the principal difference that in the zone of the laminar sub-layer ($0 \leqslant \eta \leqslant \eta_0$) we take into consideration the existence of the turbulent oscillations. Correspondingly, we take $f(R) = 1$ in Equation (2.8). In the intermediate region $\eta_0 \leqslant \eta \leqslant \eta_1$, following Karman, we give the distribution of velocities as a semi-logarithmic line, joining it smoothly with the new velocity profile in the laminar sub-layer. In the turbulent core we use the usual logarithmic velocity law in the established turbulent motion along the smooth wall. This simplified scheme does not require knowledge of the behavior of the function $f(R)$ for large R and besides, it is possible to obtain a solution in closed form.

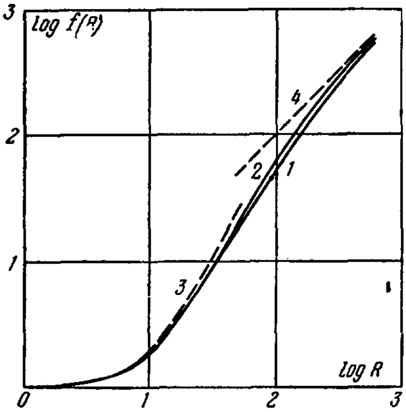

Fig. 3.

Assuming a flow of fluid in thermal equilibrium, we set

$$q = \lambda \frac{d\theta}{dy}\left(1 + P\frac{\varepsilon}{\nu}\right) = \lambda \frac{d\theta}{dy}\,[1 - P + Pf(R)] = q_w \tag{3.1}$$

or in universal coordinates

$$1 = \frac{1}{P}\frac{d\psi}{d\eta}\,[1 - P + P\,f\,(R) \tag{3.2}$$

where we set

$$\psi = \frac{\theta}{\theta_*}, \qquad \theta_* = \frac{q_w}{\rho c_p v_*} \tag{3.3}$$

The system of equations (2.7) and (3.2) allows us to obtain the following parametric solution with the quantity R as parameter:

$$\varphi = \frac{1}{2\varkappa}\int_0^R \frac{Rf'(R)+f(R)}{\sqrt{Rf^3(R)}}\,dR, \qquad \psi = \frac{1}{2\varkappa}\int_0^R \frac{Rf'(R)+f(R)}{[P^{-1}+f(R)-1]\sqrt{Rf(R)}}\,dR, \qquad \eta = \frac{1}{\varkappa}\sqrt{Rf(R)} \tag{3.4}$$

We use for small R instead of $f(R)$ the expression (2.8) and we introduce a change of variables

$$aR^2 = t^4 \tag{3.5}$$

The system of equations (3.4) leads to the following:

$$\varphi = \frac{1}{n}\int_0^t \frac{1+3t^4}{(1+t^4)^{3/2}}\,dt, \qquad \psi = \frac{1}{n}\int_0^t \frac{1+3t^4}{\left(P^{-1}+t^4\right)(1+t^4)^{1/2}}\,dt, \qquad \eta = \frac{1}{n}\,t\,(1+t^4)^{1/2} \tag{3.6}$$

For the calculation of the integrals we use an expansion of the expressions in the integrand in a power series convergent for $t < 1$. For this case, the second integral of the system (3.6), besides the algebraic part, will also contain a transcendental part $B(P^{1/4}t)$ where the symbol B stands for the function

$$B(x) = \int_0^x \frac{dx}{1+x^4} = \frac{1}{4\sqrt{2}}\ln\frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2} + \frac{1}{2\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{1-x^2} \tag{3.7}$$

tabulated by Bakhmet'ev [20].

With this form, for $t < 1$ and $n = 0.124$ we will have

$$\begin{aligned}
\varphi &= 8.06\,(t + 0.3t^5 - 0.291t^9 + 0.264t^{13} - \ldots)\\
\psi &= 8.06\,\Big[(2.5t - 0.225t^5 + 0.09t^9 - \ldots) +\\
&+ \frac{1}{P}(1.125t - 0.1625t^5) + \frac{0.08125t}{P^2} + \left(1 - \frac{2.5}{P} - \frac{1.125}{P^2}\right)P^{3/4}B(P^{1/4}t)\Big]
\end{aligned} \tag{3.8}$$

$$\eta = 8.06t\,(1+t^4)^{1/2} = 8.06\,(t + 0.5t^5 - 0.125t^9 + \ldots)$$

Using this result for the region ($0 \leqslant \eta \leqslant \eta_0$, $0 \leqslant t \leqslant t_0$) we determine the values t_0 and η_0 from the condition that the tangent to the velocity profile (3.8) at the point $\eta = \eta_0$ in the semi-logarithmic scale (ϕ, $\log \phi$) goes through the point ($\eta_1 = 30$, $\phi_1 = 13.63$) on the semi-logarithmic line

$$\varphi = 5.75 \log \eta + 5.24 \tag{3.9}$$

This corresponds to the velocity distribution of Nikuradse in this region, at a distance from the solid surface. Simple calculations indicate that for this to be true it is necessary to take $t_0 = 0.68$, $\eta_0 = 6.07$. The equation for this tangent will be

$$\begin{gathered} \varphi = 11.3 \log \eta - 3.09 \\ 6.07 \leqslant \eta \leqslant 30 \end{gathered} \tag{3.10}$$

Finally, for $\eta > 30$ the velocity profile will become the semi-logarithmic line (3.9). The proposed velocity profile is shown in Fig. 4.

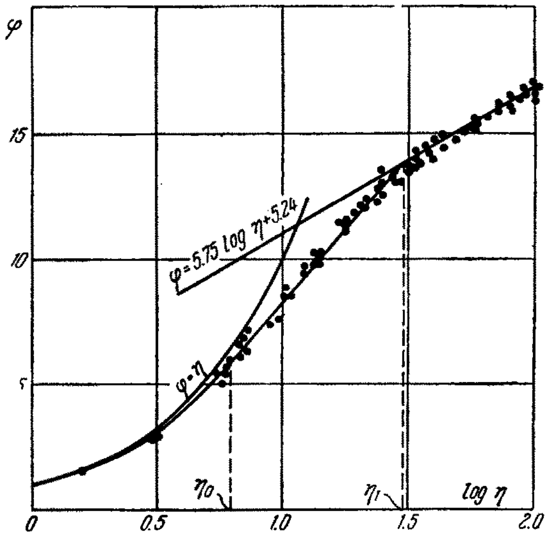

Fig. 4.

For this choice of approximate velocity profiles there correspond simple analytical expressions for the temperature profiles, consisting of the second relation of the system (3.8) for $0 \leqslant t < t_0$, $0 < \eta < \eta_0$ and the following equality for $\eta \geqslant 6.07$:

$$\begin{gathered} \psi = 11.3 \log\left(\eta + 4.98 - \frac{4.98}{P}\right) - 11.3 \log\left(11.05 - \frac{4.98}{P}\right) + \psi_0 \quad (6.07 \leqslant \eta \leqslant 30) \\ \psi = 5.75 \log \eta - 8.49 + \psi_1 \qquad (30 \leqslant \eta) \end{gathered} \tag{3.11}$$

Here by ψ_0 we will understand ψ calculated according to the second equation of (3.8) for $t = t_0 = 0.68$, and for ψ_1 the value of ψ calculated for the first equation of the system (3.11) for $\eta = \eta_1 = 30$. We exhibit the expressions for these quantities:

$$\psi_0 = 13.46 + \frac{5.976}{P} + \frac{4.453}{P^2} + \left(8.06 - \frac{20.15}{P} - \frac{9.068}{P^2} - \frac{6.549}{P^3}\right)P^{1/4}B(0.68\,P^{1/4}) \qquad (3.12)$$

$$\psi_1 = 11.3\log\left(25.09 + \frac{4.91}{P}\right) - 11.3\log\left(1.16 + \frac{4.91}{P}\right) + \psi_0 \qquad (3.13)$$

After this it is not difficult to find a new expression for the Karman function $g(P)$, which is required for the definition (1.4) of the Stanton number S_m. Taking, as before, the index m as the designation for values on the axis of the flow, we have from (1.3) and (3.3)

$$\frac{1}{S_m} = \varphi_m \psi_m \qquad (3.14)$$

For this, according to (3.9), and the last equation of the system (3.11) it is possible to set

$$\psi_m = 5.75\log\eta_m - 8.49 + \psi_1 = \varphi_m - 13.73 + \psi_1$$

Besides this, we have

$$\varphi_m = \left(\frac{2}{c_f}\right)^{1/2}, \qquad c_f = \frac{2\tau_w}{\rho U_m^2}$$

From this follows

$$\frac{1}{S_m} = \frac{2}{c_f} + \left(\frac{2}{c_f}\right)^{1/2}(\psi_1 - 13.73) \qquad (3.15)$$

Comparing expression (3.15) with (1.4) and using (3.13), we get a new expression for the Karman function

$$g(P) = \psi_1 - 13.73 = 11.3\log\left(25.09 + \frac{4.91}{P}\right) - 11.3\log\left(1.16 + \frac{4.91}{P}\right) - \qquad (3.16)$$
$$- 0.27 + \frac{5.976}{P} + \frac{4.453}{P^2} + \left(8.06 - \frac{20.15}{P} - \frac{9.068}{P^2} - \frac{6.549}{P^3}\right)P^{1/4}\,B(0.68\,P^{1/4})$$

correct for all values of Prandtl number $P \geqslant 1$.

Constructing the Nusselt number N_m, containing as characteristic temperature the temperature difference between the solid wall and the axis of the flow and assuming

$$\frac{1}{2}c_f = 0.04\,R_m^{-0.25} \qquad \left(R_m = \frac{U_m d}{\nu}\right)$$

we obtain

$$N_m = S_m R_m P = \frac{0.04 R_m{}^{3/4} P}{1 + 0.2 R_m^{-1/8} g(P)} \tag{3.17}$$

So, for example, for $R_m = 10^4$ and Prandtl number $P = 100$, from (3.17) we find $N_m = 212$. The calculation from an empirical formula of Ditus and Boelter [6]

$$N_m = 0.0254 R_m{}^{0.8} P^{0.35}$$

gives $N_m = 200$. The difference undoubtedly lies within the limits of error of the formula of Ditus and Boelter.

From Equation (3.16) comes the following asymptotic (for large P) expression of the function $g(P)$:

$$g(P) \sim 8.97 P^{3/4} + 6.26 - \frac{22.4}{P^{1/4}} + O(P^{-1}) \tag{3.18}$$

For practical calculations it is generally acceptable to make use of mean values: velocity $\bar{u}$ and temperature $\bar{\theta}$.

We observe that

$$\begin{gathered} S = \frac{1}{\bar{\varphi}\bar{\psi}} = \frac{1}{\varphi_m \psi_m} \frac{\psi_m}{\bar{\psi}} \frac{\varphi_m}{\bar{\varphi}} = S_m \frac{\psi_m}{\bar{\psi}} \frac{\varphi_m}{\bar{\varphi}} \\ c_f = \frac{2\tau_w}{\rho U_m{}^2} = 2 \frac{1}{\bar{\varphi}^2} \left(\frac{\bar{\varphi}}{\varphi_m}\right)^2, \qquad \bar{\varphi} = \frac{2\sqrt{2}}{\sqrt{\lambda}} \end{gathered} \tag{3.19}$$

Therefore, changing to the characteristic mean values, we will have

$$S = \frac{(\psi_m/\bar{\psi})\lambda}{8(\varphi_m/\bar{\varphi}) + 2\sqrt{2}\sqrt{\lambda}\, g(P)} \tag{3.20}$$

For larger values of the number P it is possible to take $\psi_m = \bar{\psi}$, and also, from the known empirical relationship of Nikuradse

$$\frac{\varphi_m}{\bar{\varphi}} = 1 + 4.08 \frac{1}{\bar{\varphi}} = 1 + 1.41\sqrt{\lambda} \tag{3.21}$$

Using these approximate relations and the asymptotic expansion (3.18), we get from (3.20) the following asymptotic relation for the Stanton number:

$$S \sim 0.04\lambda^{1/2} P^{-3/4} - (0.0124 + 0.0175\lambda^{1/2}) P^{-3/2} + 0.0985\lambda^{1/2} P^{-7/4} + O(P^{-9/4}) \tag{3.22}$$

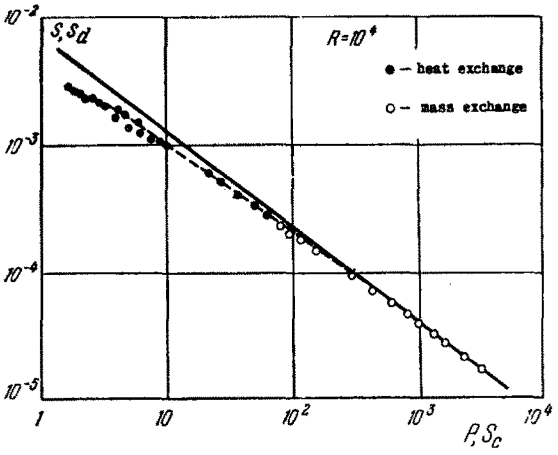

Fig. 5.

Figure 5 exhibits an empirical curve from Deissler's paper quoted above; it shows the dependence of the heat S, and consequently the Stanton diffusion number S_d, on the Prandtl number P and the Schmidt number S_c, for flow number $R = 10^4$. The full line expresses the limiting law of Deissler, which agrees with the first member of the asymptotic expansion (3.22); the first three members of the expansion give exact agreement with the calculated results. We have good agreement with the experiment in a broad range of Prandtl numbers even for $P < 10$.

In the preceding there remains untouched the problem of the value of the ratio of the coefficient of turbulent mixing of the heat content and the momentum. From the experiment of Ludwig [21], this ratio increases from the walls to the axis of the tube, and according to the experiments of Sleicher [22], on the contrary, it decreases. The contradictory experimental results have led us to take the value unity for this ratio.

The investigation of the heat transfer for small values of the Prandtl number has certain computational complications in the calculation of the mean temperature. Besides, apparently, there is a lack of adequate accurate experimental material for very small values of the Prandtl number which are necessary to establish the value s for the formula (2.20).

The preceding considerations related to the heat transfer for turbulent motion have validity also for the phenomenon of mass transfer. The difference is only that the Stanton and Nusselt numbers have to be replaced by their diffusion analogs and the Prandtl number by the Schmidt number.

BIBLIOGRAPHY

1. Prandtl, L., Bemerkung uber den Warmeubergang im Rohr. *Phys. Z.* Vol. 29, pp. 487-489, 1928.

2. Taylor, G. J., Conditions at the surface of hot body exposed to the wind. Techn. Rep. of the Adv. Committee for Aeronautics, Vol. 2, Rep. and Mem., 1916, No. 272, May, p. 423.

3. Karman, Th., Mechanische Ahnlichkeit und Turbulenz. *Nachr. Ges. Wiss. Gottingen,* 1930.

4. Karman, Th., Some aspects of the theory of turbulent motion. *Proc. Int. Congr. Appl. Mech.* Cambridge, 1934.

5. Eagle, A. and Fergusson, R., On the coefficient of heat transfer from the internal surface of tube walls. *Proc. Roy. Soc.*, A 127, p. 540, 1930.

6; Ditus, F. and Boelter, L., Heat transfer in automobile radiators of the tubular type. *University of California Publications in Engineering* Vol. 2, No. 13, p. 443.

7. Del-Nunzio, B., La distribuzione dei vortici nelle condotte chiuse in regime turbulente e laminare. *Elettrotecnica* No. 14, 1931.

8. Reichardt, H., Die Warmeubertragung in turbulenten Reibungsschichten. *Z. angew. Math. Mech.* Vol. 20, No. 6, 1940.

9. Clauser, F., Turbulent boundary layer. *Advanc. Appl. Mech.* Vol. 4, NY, 1956.

10. Levich, V. G., *Fiziko-khimicheskaia gidrodinamika (Physico-chemical Hydrodynamics)*. Gosfizmatizdat, 1959.

11. Reichardt, H., Der Einfluss der wandnahen Stromung auf den turbulenten Warmeubergang. *Mitteilung auf dem Max-Planck-Institut fur Stromungsforschung,* No. 3, 1950; Vollstandige Darstellung der turbulente Geschwindigkeitsverteilung in glatten Leitungen. *Z. angew. Math. Mech.* Vol. 31, No. 7, 1951.

12. Lin, C., Moulton, R. and Putnam, G., Mass transfer between solid wall and fluid streams. *Indust. Engrg. Chem.* Vol. 45, No. 3, 1953.

13. Elrod, H., Note of the turbulent shear stress near a wall. *J. Aero. Sci.* Vol. 24, No. 6, 1957.

14. Deissler, R., Analysis of turbulent heat transfer, mass transfer and friction in smooth tubes at high Prandtl and Schmidt Numbers. *NACA Rep.* No. 1210, 1959.

15. Martinelli, R., Heat transfer to molten metals. *Trans. Amer. Soc. Mech. Engrs.* Vol. 69, No. 8, 1947.

16. Lyon, R., Liquid metal heat-transfer coefficients. *Chem. Engrs. Progr.* Vol. 47, No. 2, 1951.

17. Kutateladze, S.S., Borishanskii, V.M., Novikov, I.I. and Fedynskii, O.S., *Zhidkometallicheskie teplonositeli (Liquid Metal Heat Conductors).* Atomizdat, 1958.

18. Loitsianskii, L.G., Turbulentnoe dvizhenie zhidkosti i vnutrenniaia zadacha (The turbulent motion of a fluid and interval flow problems). Izv. Nauchno-issl. in-ta Gidrotekhniki, Vol. 10, 1933.

19. Loitsianskii, L.G., Gipoteza local'nosti v. turbuletnom dvizhenii zhidkosti pri malichi viazkostii (The hypothesis of localness in the turbulent motion of a viscous fluid). *PMM* Vol. 22, No. 5, 1958.

20. Es'man, I.G., *Gidravlika (Hydraulics).* GONTI NKTP. Table 58, 1938.

21. Ludwieg, H., Bestimmung des Verhaltnisses der Austauschkoeffizienten fur Warme und Impuls bei turbulenten Grenzschichten. *Z. Flugwiss.* Vol. 4, Nos. 1/2, 1956.

22. Sleicher, C., Jr., Experimental velocity and temperature profiles for air in turbulent pipe flow. Paper ASME, No. 57-HT-9, 1957.

Translated by H.C.